

# Reconstruction of Tomographic Data by Markov Random Fields

Marek Zimanyi  
marek.zimanyi@fmph.uniba.sk

Department of Computer Graphics and Image Processing  
Comenius University  
Bratislava, Slovakia

## Abstract

The most important part of 3D visualization of tomographic data is an object model reconstruction. The traditional reconstruction techniques include some artefacts since the distances between slices are too big. We cannot scan the CT slices of smaller distance due to either the radiation dose or the time. We have developed a new statistical reconstruction technique based on both data modelling by Markov random fields and finding solution by Simulated annealing algorithm.

**KEYWORDS:** Reconstruction, CT, MRI, Markov Random Fields, Simulated Annealing

## 1 Introduction

CT scanner provides an information about scanned object, which is put in set of 2D projections - two dimensional pictures, slices. These slices are saved in a digital form and ready for other use.

Visualization is a very important part of CT slices processing. This scanned data visualization might be based on simple display of single pictures or on a more difficult reconstruction of a 3D object model. We are particularly interested in this second part of the CT slices visualization.

Traditional interpolation techniques contain many different artifacts (postaliasing, prealiasing, ringing ...), which arise while scanning an object (noise) or during the reconstruction itself. These artifacts depend on the size of interval the object was scanned within. It was a reason which led us towards the idea of creating a new reconstruction technique. This technique suppresses previously mentioned artifacts and uses parameters obtained from CT scanner.

The main goal of our work is to introduce a new statistical reconstruction technique which uses both Bayesian paradigm and data modelling by Markov random fields (MRF) tomographic data processing. A new consequential model is created according to the qualities of CT scanner and scanned data. Our program interpolates data with this new technique.

## 2 Traditional reconstructions and their defects

Traditional reconstruction techniques are based on the convolution of interpolation filter with sampled data. The problem arises when the slices are scanned with big distances. This problem is formulated by **Shanon** ([13] [10]): Sampling interval  $T$  has to fulfill the relation

$$T < \frac{1}{2f_{max}}, \quad (1)$$

where  $2f_{max}$  is the heightest Fourier spectral component in the sampled function and  $T = \frac{1}{f_s}$ , sampling frequency  $f_s$  is also called *Nyquist rate*.

Sampling a signal at a rate lower than postulated by Shannon leads to a very serious parasitic effect: *aliasing*.

In our case (CT slices) aliasing appears because each slice is scanned with some radiation dose and high resolution CT data are usually taken only from cadavers. In MRI tomography, scanning a sufficient number of slices for 3D reconstruction requires medically unacceptable time.

The most commonly used interpolation is linear interpolation [9] or higher interpolation techniques based on convolution filter function with sampled data (5b). However, results still suffer from staircase artifacts and false contours. A great interpolaton technique is *shape based interpolation* [6]. Although these techniques work well, removing the staircase artifacts, they can be used only for segmented objects and not for gray level tomographics.

## 3 The statistical reconstruction technique

Our technology considers parameters of CT scanner obtained from scanning the homogeneous material by this scanner. These parameters influence such final forms of CT pictures as thickness of slices or noise of apparatus.

We transform the reconstruction problem of 3D object to the 1D object reconstruction. This 1D object is a perpendicular line on slices (fig. 1a). A value of any pixel of this line is a value of the function in this point (fig. 1b, c).

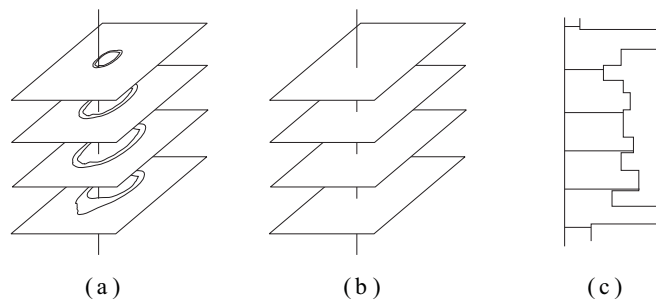


Figure 1: *CT - slices*

A principle of this proposed technique is based on finding a solution  $f^*$  with maximum likelihood (maximalization of a posteriori probability - MAP) by Simulated Annealing (fig. 2). The function  $f$  is modelled by MRF.

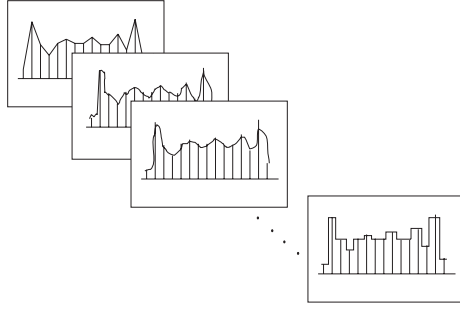


Figure 2: *Process of reconstruction*

### 3.1 Parameters of CT scanner

The parameters of CT scanner are thickness and radiation intensity. We can express these parameters by the Point Spread Function (PSF, fig. (3)). The  $x$  axis is perpendicular on CT slice and axis  $y$  is parallel to it. Scanned material is a plate with the thickness  $w$  and with the inclination  $45^\circ$  according to the plain of scanning.  $y$  axis goes through the slice and  $x$  axis is perpendicular on  $y$ , and goes through the first intersection of the material with  $y$  axis.

We examine an intensity of scanned point in  $y_0$ .

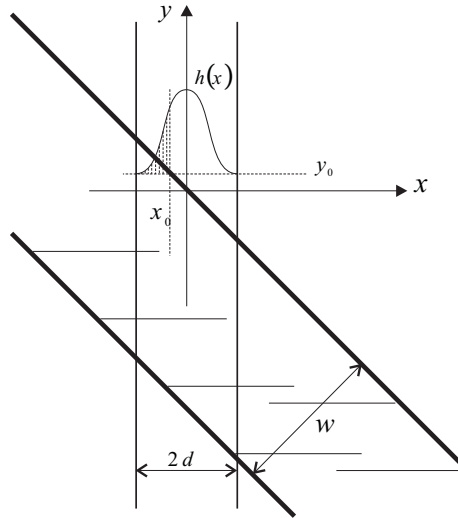


Figure 3: *The profile of a CT slice*

The value in point  $y_0$  is

$$f(y_0) = \int_{-\infty}^{+\infty} g(x, y_0) h(x) dx,$$

where  $h(x)$  is PSF and  $g(x, y_0); x \in (-\infty, \infty)$  are values of scanned material.

Since material is homogenous and scanning under the angle  $45^\circ$  it holds:

$$\begin{aligned} g(x, y_0) &= 0, & x > x_0, \\ g(x, y_0) &= 1, & x \leq x_0, \end{aligned}$$

so  $f(y_0) = \int_{-\infty}^{x_0} h(x) dx$ .

Let  $2d$  be the thickness of a slice. To find a function  $h(x)$  we need only the interval  $< -d, d >$  because values  $f(y)$  for  $y < -d$  are the same as these of an interval  $y > d$ .

Since the plate is scanned under  $45^\circ$  we get  $x_0 = -y_0$  and therefore

$$\begin{aligned} f(y_0) &= \int_{-\infty}^{-y_0} h(x) dx = \\ &= \int_{-\infty}^0 h(x) dx + \int_0^{-y_0} h(x) dx = K + H(-y_0), \end{aligned} \quad (2)$$

where  $K$  is a constant and  $H(y)$  is a primitive function according to  $h(x)$ .

Deriving (2) we get

$$f'(y) = h(y) \quad y \in < -d, d >$$

We find an interval  $< -d, d >$  from a CT slice.

But the function  $f(y)$  on interval  $< -d, d >$  is discrete and so we can substitute  $f(y)$  by the set of values  $f = \{f_0, f_1, \dots, f_m\}$ . Then

$$h_i = (f_i - f_{i-1})/d_i \quad i = 0, \dots, m,$$

where  $d_i$  is a distance between neighbor points.

We choose an approximate function (e.g. Gaussian) from these points or we use directly these points.

### 3.2 Markov and Gibbs Random Fields

Let  $\mathcal{S}$  is a discrete set of *sites* [7].

$$\mathcal{S} = \{1, \dots, m\}$$

A site often represents a point or a region in the Euclidean space as an image pixel. A rectangular lattice for a 2D image of size  $n \times n$  can be denoted by

$$\mathcal{S} = \{(i, j) | 1 \leq i, j \leq n\}$$

Let  $\mathcal{L}$  is a discrete set of *labels* (values, e.g. 256 level of grayscale). The function  $f$

$$f : \mathcal{S} \mapsto \mathcal{L} \quad f_i = f(i)$$

is called *mapping* from  $\mathcal{S}$  to  $\mathcal{L}$  and  $f = \{f_1, \dots, f_m\}$ . Function  $f$  is also called *configuration*.

When all the sites have the same labels  $\mathcal{L}$ , the set of all possible labelings is the following Cartesian product

$$L = \underbrace{\mathcal{L} \times \mathcal{L} \times \dots \times \mathcal{L}}_m = \mathcal{L}^m,$$

where  $m$  is the size of  $\mathcal{S}$ .

The sites in  $\mathcal{S}$  are related to each other via a *neighborhood system*  $N$  (fig. 4) [7]

$$N = \{N_i | \forall i \in \mathcal{S}\},$$



Figure 4: *4-neighborhood and 8-neighborhood*

where  $N$  is the set of sites neighboring  $i$ .

For a regular lattice  $\mathcal{S}$  is a neighbor set of  $i$  defined as

$$N = \{i' \in \mathcal{S} \mid d[\text{pixel}_i, \text{pixel}_{i'}] < r, i' \neq i\},$$

where  $d[A, B]$  denotes the Euclidean distance between A and B.

A *clique*  $c$  for  $\mathcal{S}$  with  $N$  is defined as a subset of sites in  $\mathcal{S}$ . A single-site clique is  $c = i$ , a pair-site clique is  $c = i, i'$ , a pair-site clique is  $c = i, i', i''$ . The collections of these cliques are

$$\begin{aligned} C_1 &= \{i \mid i \in \mathcal{S}\} \\ C_2 &= \{(i, i') \mid i' \in N, i \in \mathcal{S}\} \\ C_3 &= \{(i, i', i'') \mid i, i', i'' \in \mathcal{S} \text{ are neighbors}\} \end{aligned}$$

$F$  is a *random field* if  $F = \{F_1, F_2, \dots, F_m\}$  is a set of random variables defined on  $\mathcal{S}$  in which each variable  $F_i$  takes a value  $f_i \in \mathcal{L}$  (denotation is  $F_i = f_i$ ). We denote the probability  $P(F_i = f_i)$  as  $P(f_i)$ . And the join probability is denoted  $P(F = f) = P(F_1 = f_1, \dots, F_m = f_m)$  or abbreviated  $P(f)$ .

$F$  is said to be a *Markov random field* on  $\mathcal{S}$  with respect to a neighborhood system  $N$  if and only if

$$\begin{aligned} P(f) &> 0 \quad \forall f \in L \\ P(f_i \mid f_{\mathcal{S}-\{i\}}) &= P(f_i \mid f_N), \quad \text{Markovianity} \end{aligned} \quad (3)$$

where  $\mathcal{S} - \{i\} = \{\forall k \in \mathcal{S}; i \neq k\}$ ;  $f_N = \{f_{i'} \mid i' \in N\}$ . The Markovianity depicts the local characteristics of  $F$ . The label at a sites are depend only on neighboring sites.

$F$  is said to be a *Gibbs random field* on  $\mathcal{S}$  with respect to a neighborhood system  $N$  if and only if

$$P(f) = Z^{-1} \times e^{-\frac{1}{T}U(f)}, \quad (4)$$

where  $Z = \sum_{f \in L} e^{-\frac{1}{T}U(f)}$  is a constant called the *partition function*,  $T$  is a temperature and  $U(f)$  is the energy function. Energy is

$$U(f) = \sum_{c \in \mathcal{C}} V_c(f) = \sum_{\{i\} \in C_1} V_1(f_i) + \sum_{\{i, i'\} \in C_2} V_2(f_i, f_{i'}) + \dots \quad (5)$$

Obviously, the Gaussian distribution is a special member of this Gibbs distribution family.

The Hammer-Clifford theorem [7][4] establishes the equivalence of MRF and GRF.

**Theorem:**  $F$  is a MRF on  $\mathcal{S}$  with respect to  $N$  if and only if  $F$  is a GRF a  $\mathcal{S}$  with respect to  $N$ .

**Result:** If  $F$  is a MRF where  $f \in \mathcal{L}$  then we can express  $P(f)$  as (4).

### 3.3 Reconstruction and probability

Let  $f_{orig} \in \mathcal{L}$  is the scanning object defined on  $S_f \subset \mathcal{S}$  and  $d \in \mathcal{L}$  are data defined on  $S_d \subset \mathcal{S}$ . In general, a model can be expressed as

$$d = \varphi(f_{orig}) + e \text{ or } d_i = \varphi(f_i) + e_i \quad (6)$$

where  $\varphi(\cdot)$  is a nonlinear operator,  $e$  is a noise and  $\varphi(f_i) = \sum_{j=1}^a h(j)f_0(i-j)$ ;  $h(\cdot)$  is PSF CT-scanner or if we use the set of value  $h_j$  from CT scanner then  $h(j) = h_j$ . The operator  $\varphi$  is the convolution PSF or set of labels obtained from CT scanner (section 3.1) with the object.

Let  $f \in \mathcal{L}$  is any set of values defined  $S_f \subset \mathcal{S}$ . We want to find so  $f$  that the probability  $P(f/d)$  is maximal. So the solution is

$$f^* = \arg \max_{f \in L} P(f/d). \quad (7)$$

We express relationship between scanned data and the set of values  $f$  by *Bayes formula* [7]:

$$P(f/d) = \frac{P(d/f)P(f)}{P(d)} \quad (8)$$

$P(f/d)$  is a conditional probability (a posterior probability) where  $d$  is fixed. Probability that  $F_1 = f_1, \dots, F_m = f_m$  for  $\forall F_i \in S_f$  provided  $F'_1 = d_1, \dots, F'_m = d_m$  for  $\forall F'_i \in S_d$

$P(d/f)$  is probability that  $F = d$ ;  $\forall F_i \in S_d$  if  $F' = f$ ;  $\forall F'_i \in S_f$

$P(f)$  is probability of event  $f$ , or  $F = f$ ;  $\forall F_i \in S_f$  (a priori probability)

$P(d)$  is probability, that  $s = d$ ;  $\forall F_i \in S_d$  which is fixed for  $\forall f \in \mathcal{L}$ .

From (6) and (8) ( $e$  is independent on  $f$ ) issue

$$P(d/f) = P([\varphi(f) + e = d]/f) = P([e = d - \varphi(f)]/f) = P(e = d - \varphi(f)) \quad (9)$$

Since Gaussian distribution is a special case of Gibbs distribution, we assume  $e$  have Gaussian distribution  $e \sim \mathcal{N}(0, \sigma^2)$ .

And so

$$P(d/f) = \frac{1}{\sigma\sqrt{2\pi}} e^{-U(d/f)} \quad (10)$$

$$U(d/f) = (d - \varphi(f))^2 / 2\sigma^2 = \sum_{i=1}^m (d_i - \varphi(f_i))^2 / 2\sigma^2 \quad (11)$$

Since  $P(d)$  is fixed for  $\forall f \in L$ :

$$f^* = \arg \max_{f \in L} (P(f/d)) = \arg \max_{f \in L} (P(d/f).P(f)) \quad (12)$$

from (4):  $P(f) = Z^{-1} \times e^{-U(f)}$  where  $Z$  is a constant and from (10):  $P(f) = K^{-1} \times e^{-U(d/f)}$  where  $K$  is a constant, we get:

$$\max(P(f).P(d/f)) \iff \min(U(f).U(d/f)) \quad (13)$$

We shall express energies  $U(f)$  and  $U(d/f)$  in the following section 3.4.

### 3.4 Energy

The energy function depends on an contextual constraint. We are interested in the energies  $U(f)$  and  $U(d/f)$ , where  $U(d/f)$  is known (10) and  $U(f)$  we choose from the type of scanning function (1c). This function is a *piecewise continuous* and her set of values is *ordered*. And from these features we create the energy function  $U(f)$ .

A general contextual constraint is the *smoothness*. For spatially continuous MRFs the smoothness prior often involves derivatives.

$$U(f) = \int (f'_a(x))^2 dx, \quad (14)$$

The energy takes the minimum value of zero only if  $f$  is absolutely flat.

But we use discrete values so we use the first order approximation of the first derivative

$$U(f) = \sum_{i \in I} (f_i - f_{i-1})^2 = \sum_{c \in C} V_c(f) = \sum_{i \in S} \sum_{i' \in N_i} V_2(f_i, f_{i'}), \quad (15)$$

where  $f_i = f_a(x_i)$  a  $I$  is set of indexes where the discrete function is defined.

In general, we can express the function pair-potentials as

$$V_2(f_i, f_{i'}) = g(f_i - f_{i'}) \quad (16)$$

For the purpose of restoration, the function  $g$  is generally even

$$g(\eta) = g(-\eta) \quad (17)$$

and nondecreasing on interval  $\langle 0, \infty \rangle$

$$g(\eta) \geq 0 \quad (18)$$

When  $f$  does not contain discontinuities,  $g(\cdot)$  is usually a quadratic function  $g(\eta) = \eta^2$ . To encode piecewise smoothness,  $g$  has to satisfy a necessary condition

$$\lim_{\eta \rightarrow \infty} |g'(\eta)| = C < \infty, \quad (19)$$

where  $C \in \langle 0, \infty \rangle$  is a constant. The condition (19) allows for  $\eta \rightarrow \infty$  discontinuities. A possible choice is the truncated quadratic function

$$g(\eta) = \min\{\eta^2, \alpha\} \quad (20)$$

If we assume the additive Gauss noise,  $e \sim \mathcal{N}(0, \sigma^2)$ , the energy  $U(d/f)$  is (10).

We can express the join probability as

$$E(f) = \sum_{i \in S} (\varphi(f_i) - d_i)^2 / (2\sigma^2) + \sum_{i \in S} \sum_{i' \in N_i} g(f_i - f_{i'}),$$

or equivalently

$$E(f) = \sum_{i \in S} (\varphi(f_i) - d_i)^2 + \lambda \sum_{i \in S} \sum_{i' \in N_i} g(f_i - f_{i'}),$$

where  $\lambda = 2\sigma^2$ . We control the present of the noise in the picture by parameter,  $\lambda$ . If no noise is present then  $\lambda = 0$ .

In the 1D case where set of sites is  $\mathcal{S} = \{1, \dots, m\}$  and set of the nearest neighbors  $i$  is  $N = \{i - 1, i + 1\}$ , we can (3.4) express as

$$E(f) = \sum_{i=1}^m (\varphi(f_i) - d_i)^2 + 2\lambda \sum_{i=2}^m g(f_i - f_{i-1}), \quad (21)$$

Li [7] put for  $g(\eta)$ , when  $g(\eta)$  piecewise smoothness function, function (20) or another functions issued from *Euler equation* [3] and the conditions (17), (18) and (19)

$$\begin{aligned} g_1(\eta) &= -e^{-\eta^2} \\ g_2(\eta) &= -\frac{1}{1 + \eta^2} \\ g_3(\eta) &= \ln(1 + \eta^2) \\ g_4(\eta) &= |\eta| - \ln(1 + \eta^2) \\ g_5(\eta) &= |\eta| \end{aligned} \quad (22)$$

We have get the best results with functions  $g_4$  and  $g_5$ .

### 3.5 Minimalization

We use *Simulated Annealing* (SA) [7] for minimalization of the energy  $E(f)$ . SA simulates the physical annealing procedure in which a physical substance is melted and then slowly cooled in a search of a low energy configuration. For the escape from the local minimum to the global minimum the *Metropolis algorithm* is used. At each step the following configuration  $f'$  is randomly chosen from  $N(f)$  (the vicinity of  $f$ ), for instance, by changing one of the  $f_i$ 's into a new label  $f'_i$ .

#### Metropolis:

```

initialize  $f$ ;
repeat
    generate  $f' \in N(f)$ ;
     $\Delta E \leftarrow E(f') - E(f)$ ;
     $P = \min\{1, e^{-\Delta E/T}\}$ ;
    if  $\text{random}(0, 1) < P$  then  $F \leftarrow f'$ ;
until (equilibrium is reached)
return  $f$ 

```

SA applies a sampling algorithm, Metropolis, successively at the decreasing values of the temperature  $T$ . The starting temperature is chosen from Metropolis algorithm:  $T$  is the start temperature when number of accepted statuses is 70% - 80%. Number of iterations we can choose  $10n$  or  $5n$ , where  $n$  is a number of sites. The decreasing sequence of temporary must satisfy  $\lim_{t \rightarrow \infty} T^{(t)} = 0$ . So  $T^{(t)}$  we can choose as  $T^{(t)} = \frac{C}{\ln(1+t)}$  or  $T^{(t)} = \kappa T^{(t-1)}$



### Simulated Annealing:

```
initialize  $T$  and  $f$ ;  
repeat  
    randomly sample  $f' \in N(f)$  under  $T$ ;  
    Metropolis( $T, f$ );  
    decrease  $T$ ;  
until ( $T > T_{min}$ )  
return  $f$ 
```

## 4 Conclusions

The proposed technique offer new recourses of reconstruction object model and it seems to be a good tool to respectable reconstruction from the CT slices. However, the problematic part of these reconstructions is a choice the right energy function  $g(\eta)$  which we choose as (22). The corect choice of this energy is described in [7]. The best functions from these are  $g_4$  and  $g_5$  (fig. 5c). The statistical interpolation also allow us to manipulate with the thickness of slices, the noise and type of smoothness.



Figure 5: (a) *Original object* (b) *traditional technique* (c) *statistical reconstruction*

## References

- [1] Raj Acharya, Richard Wasserman, Jeffrey Stevens, Carlos Hinojosa, *Biomedical Imaging Modalities: A Tutorial*. Computerized Medical Imaging and Graphics 3-24, 9 May, USA 1994.
- [2] Jozef Bruna, Alois Sehr, *Celotelova vypočetni tomografie*. vyd. Avicennum Praha, 1988.
- [3] R. Courant and D. Hilbert, *Methods of Mathematical Physics*. vol. 1, Interscience Publisher Inc., New York, 1953
- [4] U. B. Desai, *Markov Random Field Models for Early Vision Problems*. U. B. Desai India, July 1993.

- [5] Andrew S. Glassner, *Principle of Digital Image Synthesis*. Morgan Kaufmann Publishers, San Francisco, California 1995
- [6] G.T. Herman, J. Zang, C.A. Bucholts, *Shape-based interpolation*. IEEE Computer Graphics and Applications, pp. 69-79, March 1992
- [7] Stan Z. Li, *Markov Random Field in Computer Vision*. Springer, 1995.
- [8] Stephen R. Marschner, Richard J. Lobb, *An Evaluation of Reconstruction Filters for Volume Rendering*. IEEE Computer Graphics and Applications, pp.100-108, 1994
- [9] J. Anthony Parker, Robert V. Kenyon, Donald E. Troxel, *Comparison of Interpolating Methods for Image Resampling*. IEEE Transactions on Medical Imaging, MI-2 pp.31-39, March 1983
- [10] John G. Proakis, D. G. Monolakis, *Digital Signal Processing*. MacMillan Publishing Company, 1989.
- [11] Milos Sramek, *Visualization of Volumetric Data by Ray Tracing*. PhD thesis, Wien, June 1996
- [12] Erich Veach, Leonidas J. Guibas, *Metropolis Light Transport*. Computer Graphics Proceedings, pp. 65-76, SIGGRAPH, Addison Wesley, Los Angeles 1997
- [13] R. Weit, *Digital Biosignal Processing*. Elsevier Science Publishers B. V. 1991