Generalized Solids of Revolution

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Abstract

Modeling technique based on sweeping is introduced. Generalizations of solids of revolution by axis bending, usage of multiple axes and usage of various metrics are presented. Generalized solids of revolution are 2.5D objects and depending on input parameters, they may contain gaps. Constrained variants offer additional modeling possibilities and easier input. Finally some examples are given.

Keywords: geometric modeling, sweeping, solids of revolution

1. Introduction

During evolution of geometric modeling, several modeling techniques were developed. In many cases, the goal was to provide not too complex technique for rapid modeling of non trivial models. Some of them are well known and widely spread. This includes such techniques as CSG, fractals, particle systems or sweeping. This paper deals with sweeping, or strictly speaking with special subset of sweeping – generalized solids of revolution.

Sweeping technique moves an object (called shape or cross section) along arbitrary curve (called path) [Grat89]. A "tail" that profile leaves while is moving (or mathematically more precisely: union along all positions) creates resulting solid. The term "*object*" was used intentionally. It can be a three dimensional solid, two dimensional filled shape or any curve (or point, if you wish unusual way of visualizing a path). Shape and orientation of object can change during sweeping. Depending on swept object, result of sweeping can be in volumetric or boundary representation (or it is just a simple patch).

While general sweeping idea gives us wide range of possibilities, it is difficult to model. Therefore special cases of sweeping are more widely spread. The simplest variant of sweeping is extrude. Two dimensional shape (usually represented by boundary curve) is swept along line and is not rotated or changed. More complex is path extrude. Path can be any curve and swept object, while not changing shape, can even stay unrotated or rotate to be perpendicular to path. Even more complex shapes can be created by lofting/skinning approach. Along path, there are several key positions where shape of swept object is defined. Between two key positions, shape is interpolated. Usual behavior is to rotate it to be perpendicular to path.

In all previous versions, closed path was an exception. Lathe (or solid of revolution) uses circular path to create three dimensional objects. Two dimensional shape is always perpendicular to path. Usual description of lathe objects uses axis instead of circular path. Center of path belongs to axis and plane created by path is perpendicular to axis.

One remark is needed at this place. Naming conventions of sweeping techniques are not consistent. One technique can have several names and there is some small variability what exactly the technique with arbitrary name allows. For example in [Smed02], above described techniques extrude, path extrude and lathe are noticed. In [Povr02], along with name lathe, also term surface of revolution is used. In this paper, the term solid of revolution is preferred for the technique earlier mentioned as lathe.

In a following text, only surface model is used (unless explicitly stated otherwise). Although solids of revolution can be defined to allow rotation by angle less than 360 degrees, in this paper only full 360 degrees rotation is discussed.

2. Generalization

Before we start to explore ways of their generalization, we will take a closer look at solids of revolution itself. There is one axis and one curve on the input side and one solid of revolution on the output side. The curve is two dimensional. In common set up, axis is one of the main axis of coordinate system (usually z) and curve is located in xz plane (often only in half plane with positive x coordinates). By rotation of plane xz around z axis, curve follows circular path. In other words, every point of curve creates circle in plane perpendicular to axis. Such planes are later referred as slicing planes or slices.

To summarize basic construction scheme (that will later be generalized): Given an axis and a curve to rotate, make slicing planes (planes perpendicular to axis). Create circle from each intersection of slicing plane and curve. Centers of circles lie on axis.

Generalization itself can be divided into several stages (next three chapters).

2.1 Axis Bending

The first generalization comes in form of bent axis. Slicing planes are no more parallel, but rather perpendicular to axis (figure 1 shows bent axis and several slicing planes). Creating circles in slicing planes follows the same rules as solids of revolution are using [Ferk00].

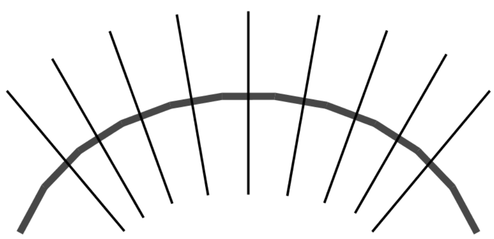

Fig. 1: Bent axis and slicing planes.

Fig. 2: Selfintersections due bending.

Allowing for bent axis, we resign to straightforward control of selfintersections. Solid of revolution can have selfintersections only if rotated curve has selfintersections. Neighboring slicing planes intersect. Area of intersection is determined by curvature of axis. Everything inside of radius of curvature in certain slicing plane cannot produce intersections due bending. Selfintersections are present at inner side of curvature (see cut of object at figure 2).

2.2 Multiple Axes

Center of circle(s) in slicing plane is a point of intersection between slicing plane and the axis. What happens when we use more than one axis? There can be more intersections of slicing plane and axes.

$$\sum_{i=0}^{n} w_i d_i(X, F_i) = c$$

Formula 1: Definition of generalized conics. X is any point of curve, F_i are foci, w_i is weight of focus i, d_i metrics of focus i and c is isovalue of curve.

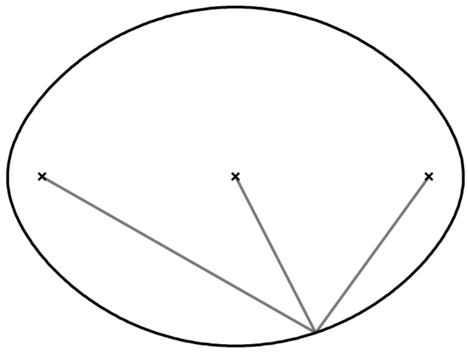

Fig. 3: Generalized conic construction.

Now every point of rotated curve, instead of curve, will follow circle with multiple centers . How does such beast look like? It is an isocurve in the slice and is called generalized conic (or n Ellipse [Seki99]). Weighted sum of distances form all foci is constant for every point of generalized conic (see formula 1 and figure 3). The constant is determined by value at intersection point between slicing plane and rotated curve [Ferk00]. Term focus has historical roots in naming of two special points inside ellipse. Later when curves with more special points arrived, term focus was adopted for them [Camp82]. More about generalized conics can be found in chapter 3.

Having several axes, slicing plane can be perpendicular to all of them only in special cases. Therefore, we need additional curve, which can determine normal of slicing plane. This curve is called leading curve and has not other reason, than determination of slicing planes position and orientation. In other words, axes have no more influence at bending. It is exclusively done by bending a leading curve.

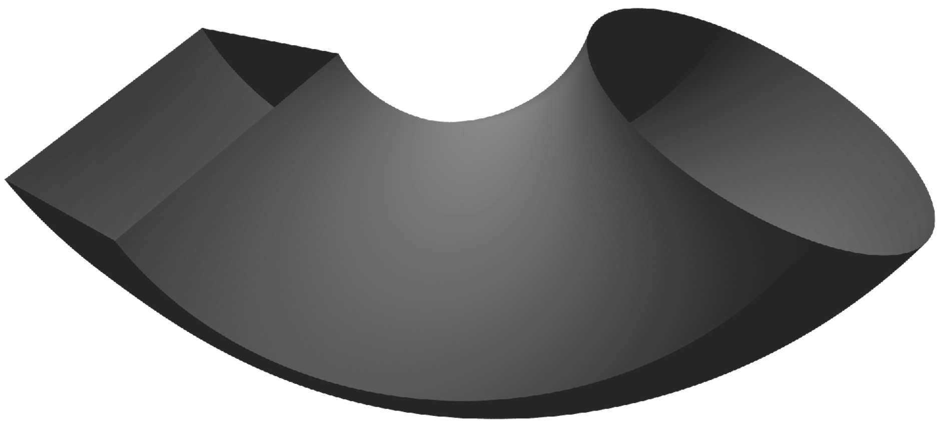

Fig. 4: Square to circle transition using blend between Manhattan and Euclidean metrics.

2.3 Various Metrics

Distance measurement for creation of generalized conics is not restricted to use of Euclidean metrics. Quite interesting results are obtained with Manhattan metrics. While Euclidean metrics creates smooth generalized conics, Manhattan one creates polygons. It is also possible to use different metrics for different foci (axes). Having two identical axes with different metrics, we can blend them together and create smooth transition between circular and square slice (figure 4). Detailed description of modeling is given in chapter 6.2.

Manhattan metrics introduces sensitivity to orientation of coordinate system (see figure 5). Rotating coordinate system, shape can change significantly. Consistent orientation of coordinate system in neighboring slices is necessary. One of possible solutions is to rotate coordinate system to align z axis to normal of slice. Translated axes x and y will create local coordinate system.

More details on usage of various metrics are given in chapter 3.

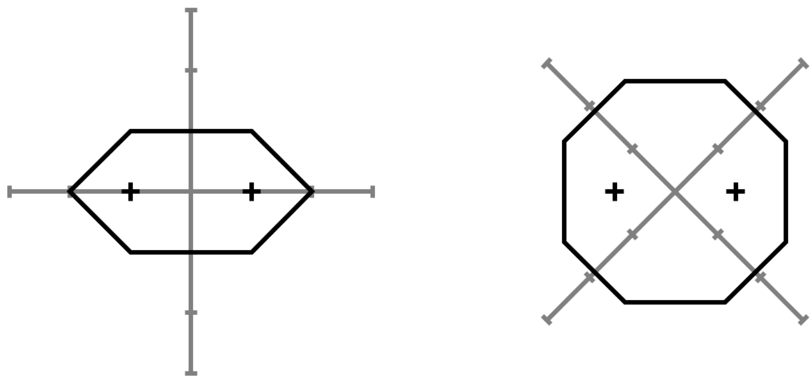

Fig. 5: Changes in shape of generalized conics for Manhattan metrics when coordinate system rotates.

2.4 Segments

By introduction of multiple axes, bending becomes more problematic. Small bending (lower than 90 degrees) is not influenced. Higher bending angles can cause occurrence of additional foci in a slice. They are caused by intersection of slicing plane with parts of axis, which are distant from expected set of foci. Due this reason it is not possible to model torus.

This restriction can be bypassed by declaration of segments. Every segment contains part of leading curve, part of each axis (void part is considered legal) and part of rotated curve. Segment is threatened as standalone generalized solid of revolution for purposes of slice creation. Segments are joined to create generalized solid of revolution. Not whole solid have to be covered by segments (this way you can create gaps in it).

As an example of segment usage, here are two pictures. Torus (figure 7) can be created using three segments. Leading curve and the only axis are identical circles, rotated curve is scaled and moved circle. Every circle is divided into three equally sized pieces (that create segments).

The second example is shell (figure 6). It is divided into nine segments way analogous to torus. Detailed description how to model shell (but using special version of generalized solid of revolution rather than segments) is given in chapter 6.

3. Generalized Conics and Their Properties

Generalized conic as defined in formula 1 is simple closed curve. For non negative weights of foci with Euclidean and Manhattan metrics, it is always convex (also for some other metrics, see [Cech02] for details). The simplest generalized conics for Euclidean metrics is circle. It has only one focus and shape is not affected with its weight. Two equally weighted foci are used to create an ellipse. Releasing restriction to equal weights, Oval of Descartes [Camp82] is created. More foci

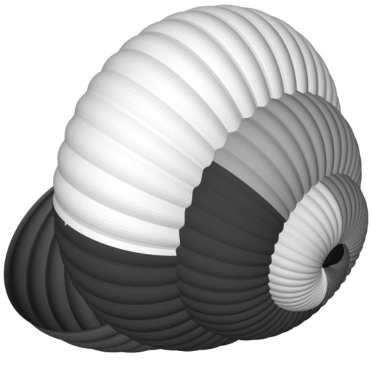

Fig. 6: Modeling of shell using segments.

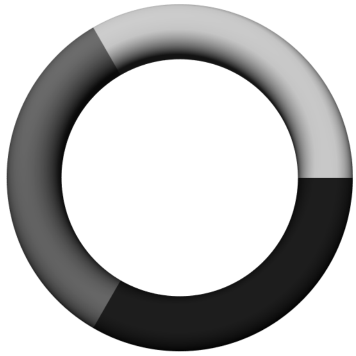

Fig. 7: Modeling of torus using segments.

give more complex curve. Generalized conic is more curved near foci. Parts of curve far from foci are similar to circle arcs.

Manhattan metrics causes angled shape of generalized conics. The simplest shape is a square. Two foci can create hexagon or octagon, depending on their relative position in coordinate system.

3.1 Reducing Number of Foci

Evaluating of generalized conics is computationally complex task. The complexity rises with number of foci. Sometimes loosing some precision in exchange of speed is acceptable tradeoff. Influence of focus to shape of generalized conic depends on several parameters. It is position of focus (relative to other foci and also distance to generalized conic is important) and its weight relative to other foci. When focus has zero weight, it does not have any influence on shape. Therefore removing such focus is all right and introduces no error.

Joining few foci with the same metrics results into simpler definition of generalized conic. Not to introduce much error, group of foci has to be chosen carefully. The main criterion is their relative proximity in contrast to distance from curve. When radius of group is significantly small in comparison with distance from curve, they can be joined without causing too much error. Resulting focus should be placed to mass center of foci group. This heuristics takes only weights and relative positions between foci in consideration. More precise heuristics (which also includes relative position of foci to curve) can be made, but computational costs are much higher.

3.2 Degenerated Generalized Conic

When intersection point has minimal value of all values in the slice, generalized conic degenerates from closed curve into different object. When using Euclidean metrics, only single point or straight line can occur. For straight line, all foci have to be collinear. Other configurations create only one point when generalized conic degenerates. In case when Manhattan metrics is used, also rectangular area can occur. Foci set have to be symmetrical by two lines parallel to both main coordinate system axes.

4. Properties of Generalized Solids of Revolution

Solids of revolution have some nice properties. In process of generalization several of them change (become optional or lost), others are left unchanged. In this chapter, we will take a closer look to those more interesting of them.

4.1 Generalized Solids of Revolution as 2.5D Object

2.5D object is three dimensional object with one nice property: every point of surface has own two dimensional coordinates (usually referred as u and v). This property is also present in generalized solids of revolution. The u coordinate is determined by position of point on generalized conic. Zero u coordinate has intersection point, the rest is marked in counterclockwise order. The v coordinate is common for all points of one generalized conic and differs for different ones.

Determining u coordinates for points of multifocal curve is not an easy task. Correct computation should ensure u coordinate values to be proportional to distances along curve. Many generalized conics are similar to circle, so first approach is to estimate u coordinate values from polar coordinates. This works perfectly for a circle, but is not so suitable for more complex (especially non symmetrical) shapes. It is also dependent on choice of start of coordinate system. The aim is to choose geometrical center of curve interior. In neighboring slices, centers positions should be correlated to avoid jerky changes in u coordinates.

To approximate distances on generalized conic with polygon, many samples are needed. To decrease computational costs, we need to approximate curve between two samples better than by straight line. Curvature of curve segment between samples can be approximated using gradients at sample points. Length of circular arc with radius equal to curvature radius is used to approximate length of segment. Testing has shown, that this approach gives 0.3% to 0.5% error for eight samples. For 50 samples (typical final mesh density) was less than one thousandth of percent.

The situation with v coordinates can be worse. With general form of generalized solids of revolution, it is not possible to just take parameter from input curve and use it as v coordinate. When leading curve parameter is chosen, several different generalized conics would have assigned the same value. Taking only rotated curve parameter, several slices can intersect with the same point of curve. Therefore, a combination of both is needed to produce correct v coordinate values. The solution is to use distance function (chapter 5.1) or slice choosing curve (chapter 5.3) instead of rotated curve. Then a parameter of distance function (or slice choosing curve) is a suitable value for v coordinate of generalized curve.

4.2 Gaps

Solids of revolution have only one reason of having gaps – rotated curve was discontinuous. Generalized solids of revolution have several more reasons why to contain gaps (gaps can also be forced manually by proper set up of slices).

4.2.1 Discontinuity of leading curve

Discontinuous leading curve causes discontinuous object. Each segment of leading curve generates standalone segment of generalized solid of revolution. However, in some special cases, usually with straight leading curve, jumping of leading curve does not cause discontinuities.

4.2.2 Tangent of leading curve does not exist

Due missing tangent of leading curve, orientation of slice cannot be determined. This causes one slice thick local discontinuity. If necessary, object can be interpolated over this gap. Blend of tangents in neighboring slices can be used to produce usable normal.

4.2.3 No foci in a slice

When slicing plane misses all axes, there are no foci and therefore no generalized conic in the slice.

4.2.4 Infinite number of intersection points in a slice

Intersections of slicing plane and rotated curve can be a point or whole part of a curve. Taking all point of intersection leads into infinite summation. There is wide range of possible solutions. One point of curve can be chosen to act as foci, or distance is measured as minimum distance from curve. Probably the simplest solution is to declare such slice illegal (in other words, to generate local discontinuity).

4.2.5 No intersection of rotated curve and slice

Where is no intersection, no generalized conic can be. Unlike previous discontinuities, this case usually does not create local discontinuity.

4.2.6 Value of distance function is too small

When value of distance function is smaller than minimum value in the slice, no generalized conic is created. When value is equal to minimum, a degenerated generalized conic occurs (see chapter 3.2).

4.2.7 Change in number of foci

Whenever change in number of foci with nonzero weights occurs, shapes of generalized conics in neighboring slices becomes non matching. Such slice creates border between two discontinuous segments.

4.2.8 Discontinuous change in weights of foci

Similar to previous case, also step change in weights of generalized conics causes discontinuity. The exception is case, when two foci share position and change in one weight is compensated in weight change of second focus.

5. Special Variants of Generalized Solids of Revolution

Allowing full range of generalizations gives us a variety in possible shapes, but modeling can become quite messy. To enable more clear inputs, we can simplify or restrict some constructions. Although we loose some shape variety, modeling becomes more powerful due clearer and more intuitive inputs.

5.1 Simplified Variant of Generalized Solids of Revolution

In fact, we don't need to have an intersection point to generate generalized conic. The same goal can be achieved using scalar value (the value that intersection point should have). Instead of rotated curve, a function (called distance function) can be used. This somehow restricts maximum number of generalized conics per slice. While number of intersections between rotated curve and slicing plane was unbounded, distance function can generate up to one generalized conic. Still, there can be no generalized conic. This time it is due fact, that value of function is less or equal to minimum of weighted distance sum in the slice.

5.2 Lofted Variant of Generalized Solids of Revolution

Avoiding unwanted foci by declaring segments is often a must. Without it, many cool looking objects would turn into ugly potatoes. Extending idea of infinity of segments, each focus can be defined per slice basis. Instead of using three dimensional curve, we can use two dimensional one. Each slice has unique number (parameter) assigned. By evaluation of leading curve for chosen parameter, a position and orientation of slice is determined. Coordinates of focus in the slice are obtained by evaluating axis at parameter. Rotated curve is also two dimensional or distance function can be used instead.

Lofted variant is very similar to lofting technique. In comparison, it does not allow such variety in swept shapes as lofting, but due existence of distance function, it enables fine level control of interpolation along the solid. Lofted variant is subset of generalized cylinders [Reha01].

5.3 Slice choosing Curve

Problem with ambiguity of v coordinate described in chapter 4.1 can be solved by using slice choosing curve instead of rotated curve. Slice choosing curve consists of two dimensional curve and slice choosing function (so in fact it is three dimensional curve). A value of slice choosing function identifies slice (value is parameter at leading curve that generates slice) and corresponding point of two dimensional curve acts as an intersection point in the slice. In contrast with usage of distance function, slice is not restricted to contain only one generalized conic. Instead of two dimensional curve, a distance function can be used.

6. Modeling Examples

Enough of theory, lets make some practical examples. Creation of pictures is made in two steps. As a first step, objects meshes are produced with package libgro (http://libgro.sourceforge.net). Visualization itself is done in Persistence of Vision raytracer (http://www.povray.org). Software package libgro is part of my master's thesis *Generalized Rotational Objects*. Thesis is available online via URL http://dw.cech.cjb.net.

6.1 Modeling a Shell

Lofted variant of generalized solid of revolution with distance function is well suited to model shell that would many snails envy. In fact, shell is only a modified cone. The only axis has both x and y coordinates constant and zero. This way it is identical to leading curve, that goes directly through the center of cone. Leading curve has spiral shape. Distance function is simple linear falloff from one to zero with three percent modulation to make rings. The exact definition is summed in table 1 and shown in figure 8. Resulting image is shown in figure 9.

Curve/function	*Definition*
Leading curve	$x=t\cos(8\pi t)$ $y=t\sin(8\pi t)$ $z=2(1-\cos(\frac{1}{2}\pi t))$
Axis	$x=0$ $y=0$
Distance function	$t(0.97+0.03\sin(173.2\pi t))$

Table 1: Definition of shell (all curves and functions have range of definition <0, 1>).

6.2 Square to Circle Transition Using Two Metrics

Both square and circle can be created using only one axis (with Manhattan metrics in case of square and Euclidean in case of circle). Unfortunately, one axis cannot have more than one metrics assigned. We need to use little trick – two spatially identical axes, that differs only with metrics and weight. Along axes, a linear blend of weights is created. While "Manhattan" one's weight fades out (from one to zero), "Euclidean" one's weight fades in (from zero to one). Sum of both weights is one in each point of axes. Rest of definition is simple. Leading curve is identical to axes and distance function is constant (any positive value does the job fine). The final image is shown in figure 4.

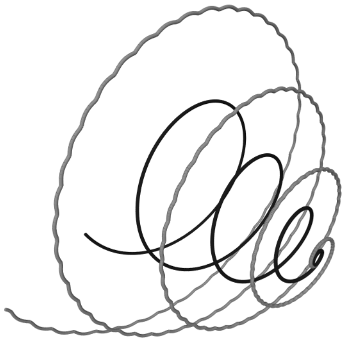

Fig. 8: Definition of shell. Leading curve and axis (identical) are black, distance function (visualized as rotated curve) is gray.

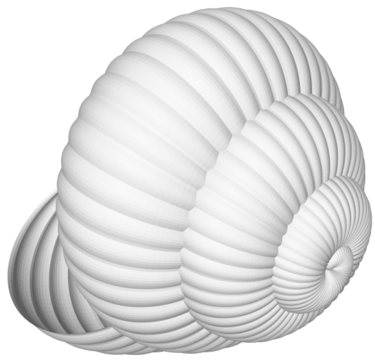

Fig. 9: A shell.

7. Conclusion

Generalized solids of revolution introduce new possibilities to modeling. Although the process of generalization brings some difficulties, most of them can be avoided by carefully chosen parameters (including proper division into segments). Special variants, in contrast to general form, suffer less from modeling problems. They offer some additional functionality in exchange of some modeling possibilities.

Generalized solids of revolution bring new ways of modeling. It offers wide range of possibilities in copying of reality and also in computer arts. In fact, our new method combines – as the shell example shows – the desirable properties of modeling, special (procedural) modeling and functional representation (F-rep).

There are still open ways of research (for example using axes with negative weights).

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