Continuity of Bézier patches

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Abstract

The paper is concerned about the question of smooth glueing of triangular Bézier patches. In the begining polar forms are briefly explained. After that they are applied on parametric continuity. We'll obtain a geometric interpretation of C^1 and C^2 smoothly joined Bézier patches.

In the next part we'll deal in geometric continuity. We will show relations between derivatives of two maps which are forced by the condition of their geometric continuity. In the place of shape parameters matrices occur. The rest of the paper is devoted to investigating of this matrices and reparametrization functions. It seems that the matrices are not arbitrary, but very strong conditions are put on them.

1 Basic definitions

1.1 Bézier patches

Let points $t_0, t_1, t_2 \in E^2$ are non-colinear. Then for each $u \in E^2$ there exist unique triplet of numbers $\lambda_0(u), \lambda_1(u), \lambda_2(u) \in R$, satisfying

$$u = \lambda_0(u)t_0 + \lambda_1(u)t_1 + \lambda_2(u)t_2; \qquad \lambda_0(u) + \lambda_1(u) + \lambda_2(u) = 1.$$

The numbers $\lambda_0(u), \lambda_1(u), \lambda_2(u)$ are called **barycentric coordinates of the point u** with respect to the triangle $t_0t_1t_2$.

In next let's suppose that the points t_0, t_1, t_2 are fixly given.

Polynomial $B^{\triangle}_{ijk} : E^2 \to R$ defined

$$B^{\triangle}_{ijk}(u) = \binom{n}{ijk} \lambda_0^i(u)\lambda_1^j(u)\lambda_2^k(u); \quad i + j + k = n$$

is called **Bernsten polynomial**. Notice, that all Bernstein polynomials of degree n (i.e. $i + j + k = n$) create basis of the space of polynomials of degree at most n.

Let $B : E^2 \to E^d$ is defined :

$$B(u) = \sum_{i+j+k=n} B^{\triangle}_{ijk}(u) b_{ijk}; \quad b_{ijk} \text{ are points from } E^d$$

The restriction of this map to $\triangle t_0 t_1 t_2$ is called **triangular Bézier patch of degree n**. Points b_{ijk} are **control points** a create **a control net**.

1.2 Polar forms

A map $f : M \to N$ is **linear** iff it preserves linear combinations, i. e.

$$f(\sum \alpha_i u_i) = \sum \alpha_i f(u_i); \quad u_i \in M,\ \alpha_i \in R$$

$f : M \to N$ is **affine** iff it preserves affine combinations of points :

$$f(\sum \alpha_i j_i) = \sum \alpha_i f(u_i); \quad u_i \in M,\ \sum \alpha_i = 1,\ \alpha_i \in R$$

$f : (M)^n \to N$ is **multilinear (multiaffine)** iff f is linear (affine) in each argument when the others are fixed.
$f : (M)^n \to N$ is **symmetric** iff its value doesn't depends on the order of arguments. It means

$$f(u_{\pi_1}, u_{\pi_2}, \dots u_{\pi_n}) = f(u_1, u_2, \dots u_n)$$

where $(\pi_1, \pi_2, \dots \pi_n)$ is a permutation of the set $(1, 2, \dots n)$.

Now we can approach a definition of polar forms.

Let F is polynomial $E^2 \to E^d$ of degree n. Then there exists unique map $f : (E^2)^n \to E^d$ which is multiaffine, symmetric and has a diagonal property, i. e. $f(u, \dots u) = F(u)$. This map is called **(affine) blossom (polar form) of F**.

For our cogitations linear blossoms are more usefull. For this purpose we must present this two special insertions E^2 to E^3:

$$u = (u_1, u_2) \;\Rightarrow\; \hat{u} = (u_1, u_2, 0)$$
$$u = (u_1, u_2) \;\Rightarrow\; \bar{u} = (u_1, u_2, 1)$$

Linear blossom (polar form) of the polynomial $F : E^2 \to E^3$ is a map $f_* : (E^3)^n \to E^d$, which is multilinear, symmetric and has a diagonal property, i. e. $f_*(\bar{u}, \bar{u}, \dots \bar{u}) = F(u)$. Such map always exist and is unique.

Linear and affine blossoms of the same polynomial F satisfy

$$f_*(\bar{u}_1, \bar{u}_2, \dots \bar{u}_n) = f(u_1, u_2, \dots u_n); \qquad u_i \in E^2$$

2 Parametric continuity

Maps $F, G : E^2 \rightarrow E^d$ are **parametric continuous of the order q** in $u \in E^2$ if their directional derivatives in u up to order q are the same.

The theory of polar forms can be applied on parametric continuity of surfaces. But first let's mention their two important properties:

Let F be a Bézier patch of degree n with control points b_{ijk} for $i + j + k = n$ with respect to $\triangle t_0 t_1 t_2$ and let f be its polar form. Then

$$b_{ijk} = f(t_0^i t_1^j t_2^k); \qquad i + j + k = n$$

Furthermore if we denote $D_{\xi_1 \xi_2 \ldots \xi_q} F(u)$ q-th directional derivative of F in u with respect to vectors $\xi_1 \ldots \xi_q$, then the linear blossom of $F : E^2 \rightarrow E^d$ satisfy relation

$$D_{\xi_1 \xi_2 \ldots \xi_q} F(u) = \frac{n!}{(n-q)!} f_*(\bar{u}^{n-q} \hat{\xi}_1 \hat{\xi}_2 \ldots \hat{\xi}_q) \tag{1}$$

Proofs of this properties you can find for example in [3].

We started to use a simpler and lucider multiplicative notation:
Instead of $f(u_1, u_2, \ldots u_n)$ we are going to write $f(u_1 u_2 \ldots u_n)$. By the entry $f(u^{n-k} u_1 u_2 \ldots u_k)$ we want to say that argument u occurs $(n-k)$ times there.

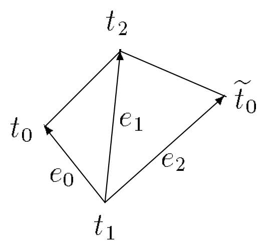
Therefore, if we want to find out a value of a derivative of a polynomial, we don't need to derivate, but it is enough to look at a certain value of its blossom.

From the previous theorem we get conditions for parametric continuity of two patches:

Let $F, G : E^2 \rightarrow E^d$ are triangular Bézier patches and let u be a point from E^2. Then according to (1) F and G are continuous in u of q-th order iff

$$f_*(\bar{u}^{n-i} \hat{\xi}_1 \ldots \hat{\xi}_i) = g_*(\bar{u}^{n-i} \hat{\xi}_1 \ldots \hat{\xi}_i); \quad i = 0, \ldots q$$

3 Geometric interpretation



$$e_0 = t_0 - t_1$$
$$e_1 = t_2 - t_1$$
$$e_2 = \widetilde{t}_0 - t_1$$

Figure 1: Mapped space

Let's denote barycentric coordinates of $\widetilde{t}_0$ with respect to $\triangle t_0t_1t_2$ as $\lambda_0, \lambda_1, \lambda_2 : \widetilde{t}_0 = \lambda_0 t_0 + \lambda_1 t_1 + \lambda_2 t_2$ where $\lambda_0 > 0$. Let Bézier patches F, G map this point cofiguration in this way:

$$F : \triangle t_0t_1t_2 \to E^2$$
$$G : \triangle \widetilde{t}_0t_2t_1 \to E^2$$

Further let b_{ijk} are control points of the patch F and $\widetilde{b}_{ijk}$ are control points of the patch G. Let's investigate a continuity of this two patches along the common edge t_1t_2. If we suppose C^0 continuity, it is obvious, that derivatives in $u \in t_1t_2$ with respect to e_1 are the same – it's the same Bézier curve. So that it is enough to find only one more direction already, which is lineary independent with e_1, with respect to which the maps F, G poses the same derivative. Then the C^1 continuity is satisfied because of the fact that $D_\xi F(u)$ is linear in ξ for a fixed u and so the other directional derivatives can be composed from the mentioned two.

We can write :

$$\begin{aligned}
\widetilde{t}_0 - t_1 &= \lambda_0(t_0 - t_1) + \lambda_2(t_2 - t_1)\\
e_2 &= \lambda_0 e_0 + \lambda_2 e_1\\
D_{e_2}F(u) &= \lambda_0 D_{e_0}F(u) + \lambda_2 D_{e_1}F(u)\\
D_{e_2}G(u) &= D_{e_2}F(u)\\
D_{e_2}G(u) &= \lambda_0 D_{e_0}F(u) + \lambda_2 D_{e_1}F(u)\\
g_*(\bar{u}^{n-1}\hat{e}_2) &= \lambda_0 f_*(\bar{u}^{n-1}\hat{e}_0) + \lambda_2 f_*(\bar{u}^{n-1}\hat{e}_1)\\
&\dots\\
g(\widetilde{t}_0 t_1^i t_2^j) &= \lambda_0 f(t_0 t_1^i t_2^j) + \lambda_1 f(t_1^{i+1} t_2^j) + \lambda_2 f(t_1^i t_2^{j+1})\\
\widetilde{b}_{1ij} &= \lambda_0 b_{1ij} + \lambda_1 b_{0i+1j} + \lambda_2 b_{0ij+1}; \qquad i + j = n - 1,\ u \in t_1t_2
\end{aligned}$$

And thanks to the blossoming we have simply obtained well-known geometric interpretation of the C^1 continuity:

Bézier patches $F : \triangle t_0t_1t_2 \to E^d, G : \triangle t_1\widetilde{t}_0t_2$ are C^1 smoothly joined along the common boundary iff the control points $b_{1ij}, b_{0i+1j}, \widetilde{b}_{1ij}, b_{0ij+1}$ lie in the same plane and create an affine image of the quadrangle $t_0t_1\widetilde{t}_0t_2$ (look at fig. 2).

After similar cogitations we can obtain not so known geometric interpretation of C^2 continuity:

Bézier patches $F : \triangle t_0t_1t_2 \to E^d, G : \triangle t_1\widetilde{t}_0t_2$ are C^2 smoothly joined along the common boundary iff moreover there exist points $d_i;\ i = 1, \dots n-1$ such that verteces $b_{2ij}, b_{1i+1j}, d_i, b_{1ij+1}$ lie in the same plane and create an affine image of the quadrangle $t_0t_1\widetilde{t}_0t_2$ and also points $d_i, \widetilde{b}_{1i+1j}, \widetilde{b}_{2jk}, \widetilde{b}_{1ij+1}$ are coplanar and create an affine image of the quadrangle $t_0t_1\widetilde{t}_0t_2$ too (fig. 3).

4 Geometric continuity

Let $F, G : E^2 \to E^d$ are maps. Let $u \in E^2$. Then F, G are in u **geometric continuous** of q-th order iff there exist such parametrization of surfaces F, G, that

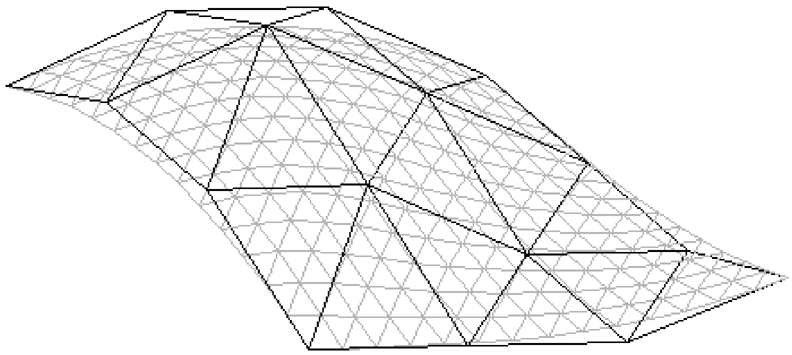


Figure 2: C^1 continuity

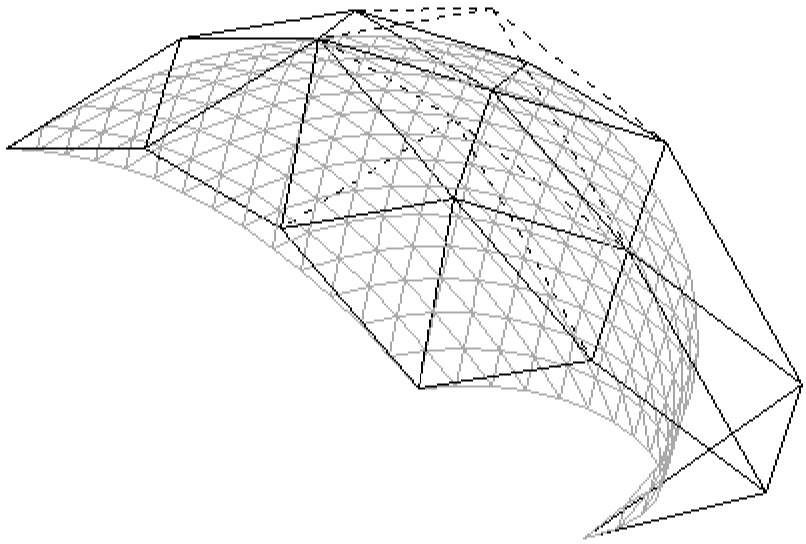


Figure 3: C^2 continuity

they are parametric continuos of q-th order.

It means:

$$F = F(x, y) \text{ i.e. is a function of parameters } x, y$$
$$G = G(s, t) \ .$$

Let's try to reparametrize the map F and express it by s, t. So x and y will be written as functions: $x = x(s, t),\ y = y(s, t)$. Let both maps are now expressed in the way, that they are parametric continuous in $u = (s_0, t_0)$:

Continuity of first degree:

$$F_s(u) = G_s(u) \quad \wedge \quad F_t(u) = G_t(u)$$

For continuity of second degree moreover

$$F_{ss}(u) = G_{ss}(u) \quad \wedge \quad F_{st}(u) = G_{st}(u) \quad \wedge \quad F_{tt}(u) = G_{tt}(u)$$

But F has to be derivated as a compound function. Let's denote

$$\begin{aligned} a_{ij}(u) &= \frac{\partial^{i+j}}{\partial s^i \partial t^j} x(u) \\ b_{ij}(u) &= \frac{\partial^{i+j}}{\partial s^i \partial t^j} y(u); \quad u \in E^2 \end{aligned}$$

Then the relation between derivatives of F and G with respect to the original parameters can be expressed by connection matrices

$$\begin{pmatrix} G_s \\ G_t \end{pmatrix} = \begin{pmatrix} a_{10} & b_{10} \\ a_{01} & b_{01} \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} G_{ss} \\ G_{st} \\ G_{tt} \end{pmatrix} = \begin{pmatrix} a_{20} & b_{20} \\ a_{11} & b_{11} \\ a_{02} & b_{02} \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix} + \begin{pmatrix} a_{10}^2 & 2a_{10}b_{10} & b_{10}^2 \\ a_{10}a_{01} & a_{10}b_{01} + a_{01}b_{10} & b_{10}b_{01} \\ a_{01}^2 & 2a_{01}b_{01} & b_{01}^2 \end{pmatrix} \begin{pmatrix} F_{xx} \\ F_{xy} \\ F_{yy} \end{pmatrix} \tag{3}$$

Let G' is the matrix of first derivatives of G, G'' is the matrix of its second derivatives and so on. Similary let F' is the matrix of first derivatives of F, F'' is the matrix of its second derivatives and so on. Then the relations (2), (3) and other can be written in a simpler way:

$$\begin{aligned} G' &= \Omega_{11} F' \\ G'' &= \Omega_{12} F' + \Omega_{22} F'' \\ G''' &= \Omega_{13} F' + \Omega_{23} F'' + \Omega_{33} F''' \\ &\dots \end{aligned} \tag{4}$$

Surfaces F and G are in u geometric continuous of q-th degree iff there exist matrices

$$\Omega_{ij}; \quad j = 1 \dots q, \ i = 1 \dots j$$

satisfying the mentioned relations.

We observe that the relations for the geometric continuity of surfaces are very similar to the ones for the geometric continuity of curves. But as shape parameters there are matrices Ω_{ij} instead of real numbers.

5 Geometric continuity of Bézier patches

Now we are going to apply the results of the previous section on the Bézier patches.

Let Bézier patches F, G are defined on the quadrangle $t_0 t_1 \tilde{t}_1 t_2$ like this:

$$F : \triangle t_0 t_1 t_2 \to E^2$$
$$G : \triangle \tilde{t}_0 t_2 t_1 \to E^2$$

Let F and G are regularly aparametrized in this way:

$$F = F(r, s)$$
$$G = G(s, t)$$

where parameter r raises in the direction e_0, s raises in the direction e_1 and t in the direction e_2 (look fig.1). We suppose C^0 continuity : $F(0, s) = G(s, 0)$. In the begining we require geometric continuity of the first order of the patches F and G along the common boundary $t_1 t_2$. It means that for all u from this edge there exist a matrix Ω_{11}, satisfying (2). In this special case it can be written:

$$\begin{pmatrix} D_{e_1} G \\ D_{e_2} G \end{pmatrix} = \begin{pmatrix} a_{10} & b_{10} \\ a_{01} & b_{01} \end{pmatrix} \begin{pmatrix} D_{e_0} F \\ D_{e_1} F \end{pmatrix}$$

where a_{ij}, b_{ij} are functions of the point u.

Thanks to the fact that parameter s is common for both F and G, it holds $D_{e_1} G(u) = D_{e_1} F(u)$ for all $u \in t_1 t_2$ and so $a_{10} = 0$ and $b_{10} = 1$ constantly.

Let $a = a_{01}$ $b = b_{01}$ for simplicity. Then :

$$\Omega_{11} = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}; \qquad a = a(u) \quad b = b(u)$$

The result can be geometricaly interpreted : Bézier patches are in u geometric continuous of first degree iff they have in u the same tangent plane.

Let's continue by investigating of geometric continuity of second degree of two Bézier patches. Besides the matrix Ω_{11}, with the specified form, the matrices Ω_{22} and Ω_{12} must exist, such that

$$G'' = \Omega_{12} F' + \Omega_{22} F''$$

If we look at (3), we can notice, that all elements of the matrix Ω_{22} are already known, because it is composed only of coeficients occured in Ω_{11}.

And as for Ω_{12}, it's possible to proceed like in the case of Ω_{11}. Because the same parameter and the derivative of the same curve is considered, it holds

$$D_{e_1 e_1} G(u) = D_{e_1 e_1} F(u) \;\Rightarrow\; a_{20} = b_{20} = 0$$

It will be a bit more complicated for the combined derivative:

$$\begin{aligned} D_{e_1e_2}G(u) &= D_{e_1}(D_{e_2}G(u)) \\ &= D_{e_1}(aD_{e_0}F(u) + bD_{e_1}F(u)) \\ &= aD_{e_1}(D_{e_0}F(u)) + bD_{e_1}(D_{e_1}F(u)) \\ &= aD_{e_0e_1}F(u) + bD_{e_1e_1}F(u) \end{aligned}$$

and after comparing with (3) we obtain $a_{11} = b_{11} = 0$ constantly for all $u \in t_1t_2$.

The result of this observations is the next equation :

$$\begin{pmatrix} D_{e_1e_1}G \\ D_{e_1e_2}G \\ D_{e_2e_2}G \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ c & d \end{pmatrix} \begin{pmatrix} D_{e_0}F \\ D_{e_1}F \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & a & b \\ a^2 & 2ab & b^2 \end{pmatrix} \begin{pmatrix} D_{e_0e_0}F \\ D_{e_0e_1}F \\ D_{e_1e_1}F \end{pmatrix}$$

We could continue similary. After the small cogitation above the equation for the q-th derivative

$$G^{(q)} \;=\; \Omega_{1q}F' + \Omega_{2q}F'' + \cdots + \Omega_{qq}F^{(q)} \tag{5}$$

we realize that if the previous derivatives has been investigated, new coeficients appear only in matix Ω_{1q} – the q-th derivatives of the reparametrization functions are only there. Let's look at it more closely.

Let the reparametrization functions x and y are polynomials of degree m (look [2]) :

$$x(u,v) = \sum_{i+j\le m} \alpha_{ij}u^iv^j \quad \wedge \quad y(u,v) = \sum_{i+j\le m} \beta_{ij}u^iv^j$$

From the matrix Ω_{12} it follows $x_{uv} = a_{11} = 0$ and also $y_{uv} = b_{11} = 0$. It means that nor $x(u,v)$ neither $y(u,v)$ contains combined member. Furthermore from Ω_{11} it is $x_u = a_{10} = 0$, i. e. x doesn't depend on parameter u. In the case of y it is a bit different: $y_u = b_{10} = 1$ constantly along the whole boundary t_1t_2. It means that $y(u,v)$ contains u only as a linear member witch coeficient 1. The result can be written:

$$x(u,v) = x(v) = \sum_{i=0}^{n} \alpha_i v^i$$

$$y(u,v) = u + \sum_{i=0}^{n} \beta_i v^i$$

And then we obtain more specified form of Ω_{1j} :

$$\Omega_{1q} = \begin{pmatrix} a_{q0} & b_{q0} \\ \cdots & \\ a_{1q-1} & b_{1q-1} \\ a_{0q} & b_{0q} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \cdots & \\ 0 & 0 \\ a_{q0} & b_{q0} \end{pmatrix}$$

Only last two members of Ω_{1q} can get an arbitrary value.

Let's look at the result. It seems that if the conditions of GC^{q-1} smooth joining of the patches F and G has been formulated and we want to reach smoothnes of q-th order, it can be determined by only two shape parameters in the matrix Ω_{1q}. There's no possibility to manipulate with other matrices occured in the relation between $G^{(q)}$ and $F^{(q)}$ (look (5)). So the geometric continuity of curves and surfaces are very similar not only because of the form of the equation system (4) but also because of the constraints for the work with shape parameters.

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